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LOGARITHMIC DERIVATIVES OF THETA FUNCTIONS

BY

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ABSTRACT

We provide two new proofs of the identity

$$\sum_{n=0}^{\infty} \delta(3n+1)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{3n})^3}{(1-x^n)}$$

where $\delta(n) = d_1(n) - d_2(n)$ and $d_i(n)$ is the number of divisors of n congruent to $i \bmod 3$. Furthermore, we express the number of solutions of the Diophantine equation $x^2 + 3y^2 = N$ in terms of $\delta(N)$.

1. Introduction

The theory of theta functions with characteristics

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau) = \sum_{N\in Z^g} exp(2\pi i)\Big(\frac{1}{2}\Big(N+\frac{\epsilon}{2}\Big)^t\tau\Big(N+\frac{\epsilon}{2}\Big) + \Big(N+\frac{\epsilon}{2}\Big)\Big(\zeta+\frac{\epsilon'}{2}\Big)\Big)$$

defined for $\zeta \in C^g$, τ a symmetric $g \times g$ matrix with positive definite imaginary part and ϵ, ϵ' vectors in R^g, plays a central role in the theory of abelian varieties, compact Riemann surfaces and, when $g = 1$, combinatorial number theory. As a function of ζ with τ fixed, they are the bulding blocks of the theory of multiply periodic functions and, as a function of τ (usually with $\zeta = 0$), the building blocks of the theory of modular forms. The modular theory (τ-theory), especially when $g > 1$, is deeply elegant and useful and is much more difficult than the ζ-theory. Many combinatorial number theoretic properties can be deduced from identities whose proofs are very simple after appealing to

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the theory of modular forms. Sometimes, these proofs do not give any insight into the question but sometimes they do. Since the ζ-theory is much more elementary, it seems reasonable to ask whether the theory of modular forms is really necessary in a particular problem or can one get the result from ζ-theory.

In the book [FK], these theories are exposed in an attempt to show that there is a general theory which handles questions in combinatorial number theory. The reader is advised to consult this book and the literature references cited there. In [F1] it was shown that by defining

$$\delta(n) = d_1(n) - d_2(n)$$

where $d_i(n)$ is the number of divisors of n congruent to $i \bmod 3$, one gets a relation between the multiplicative function $\delta(n)$ and the function $\sigma(n)$, where $\sigma(n)$ is the sum of the divisors of n. As a consequence the identity

$$\sum_{n=0}^{\infty} \sigma(3n+2)x^n = 3\Big(\sum_{n=0}^{\infty} \delta(3n+1)x^n\Big)^2$$

is proven.

In this paper we give new proofs of the result

$$\sum_{n=0}^{\infty} \delta(3n+1)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{3n})^3}{(1-x^n)}.$$

One of our proofs will use an identity which requires the modular theory. We present this identity because we feel it is natural and pretty. The second proof we offer uses neither this identity nor any modular considerations. We think this is worth showing in its own right, but more importantly it shows how techniques of Godin can be used to solve problems of this sort as well as the solution to the Diophantine equation given at the end of the paper.

Eta-products and eta-quotients $\eta^N(N\tau)/\eta(\tau)$ have been considered by many authors. See [DKK] and [M], for example, and [BB] or [S] for other proofs of this identity.

We consider in detail the case $N = 3$. In the variable $x = exp(2\pi i\tau)$ we have

$$\frac{\eta^3(3\tau)}{\eta(\tau)} = x^{1/3} \prod_{n=1}^{\infty} \frac{(1-x^{3n})^3}{(1-x^n)} = g(x).$$

In order to have a better picture of this function we begin by changing variables and let $x = y^3$, so that in the variable y we have

$$\frac{\eta^3(3\tau)}{\eta(\tau)} = y \prod_{n=1}^{\infty} \frac{(1-y^{9n})^3}{1-y^{3n}} = \sum_{n=0}^{\infty} g_{3n+1} y^{3n+1}.$$

It thus follows that

$$\prod_{n=1}^{\infty} \frac{(1-x^{3n})^3}{1-x^n} = \sum_{n=0}^{\infty} g_{3n+1} x^n.$$

If one computes g_{3n+1} for the first few values of n we see that

$$g_1 = 1,\ g_4 = 1,\ g_7 = 2,\ g_{10} = 0,\ g_{13} = 2,\ g_{16} = 1,\ \dots$$

so that one begins to suspect that $g_{3n+1} = \delta(3n+1)$.

2. A related function

In order to get a grip on the coefficients we are seeking, we define in this section a new function $\prod_{n=1}^{\infty}(1-x^n)^3/(1-x^{3n})$ (see Table I in [M]). Let us denote this function by $f(x)$. In this section we compute the power series coefficients of $f(x)$ and write

$$f(x) = 1 - 3\sum_{n=1}^{\infty} f_n x^n.$$

LEMMA 1: *If $m \geq 0$ is an integer, then $f_{3m+1} = g_{3m+1}$ and $f_{3m} = -2\delta(m)$.*

Proof: We begin by recalling Jacobi's identity

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{\frac{k^2+k}{2}}.$$

It is clear from the identity that

$$f(x) = \frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{n^2+n}{2}}}{\prod_{n=1}^{\infty}(1-x^{3n})}.$$

By summing on congruence classes mod 3, the numerator of the above fraction is easily seen to be equal to

$$\sum_{n=0}^{\infty}(-1)^n(6n+1)x^{\frac{9n^2+3n}{2}} + \sum_{n=0}^{\infty}(-1)^{n+1}(6n+3)x^{\frac{9n^2+9n+2}{2}} + \sum_{n=0}^{\infty}(-1)^n(6n+5)x^{\frac{9n^2+15n+6}{2}},$$

which we shall rewrite as

$$-3x\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{9\frac{n^2+n}{2}} + \sum_{n=0}^{\infty}(-1)^n(6n+1)x^{3\frac{3n^2+n}{2}} + \sum_{n=0}^{\infty}(-1)^n(6n+5)x^{\frac{9n^2+15n+6}{2}}.$$

The last sum can be rewritten as

$$\sum_{n=0}^{\infty}(-1)^n(6n+5)x^{\frac{9n^2+15n+6}{2}} = \sum_{n=1}^{\infty}(-1)^{n-1}(6n-1)x^{\frac{9n^2-3n}{2}} = \sum_{n=-1}^{-\infty}(-1)^n(6n+1)x^{\frac{9n^2+3n}{2}},$$

hence the expression for $f(x)$ now becomes

$$f(x) = -3x\prod_{n=1}^{\infty}\frac{(1-x^{9n})^3}{1-x^{3n}} + \frac{\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)x^{3\frac{3n^2+n}{2}}}{\prod_{n=1}^{\infty}(1-x^{3n})}.$$

Now we use the following formula, which was proved in [F1]:

$$\frac{\theta'\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)} = 2\pi i\left(\frac{1}{6}+\sum_{n=1}^{\infty}\delta(n)y^n\right).$$

Expanding the numerator of the left-hand side in series and using the Jacobi triple product formula for the denominator, we get

$$\frac{\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)y^{\frac{3n^2+n}{2}}}{\prod_{n=1}^{\infty}(1-y^n)} = 1+6\sum_{n=1}^{\infty}\delta(n)y^n.$$

After replacing y by $y^9 = x^3$ in the last equality, we finally obtain

$$f(x) = -3\sum_{n=0}^{\infty}g_{3n+1}x^{3n+1} + 1 + 6\sum_{n=1}^{\infty}\delta(n)x^{3n}.$$

It is obvious from this last representation that we may write

$$f(x) = 1-3\sum_{n=1}^{\infty}f_nx^n$$

in the form we suggested at the beginning of the section. This concludes the proof of the lemma. ∎

3. Relationships to the σ-function

In this section we shall give the modular proof of the assertion that $\delta(3n+1) = g_{3n+1}$. In [F2], the following identity is proven:

$$x\prod_{n=1}^{\infty}\frac{(1-x^{9n})^3(1-x^n)^3}{(1-x^{3n})^2} = \sum_{n=0}^{\infty}\sigma(3n+1)x^{3n+1} - \sigma(3n+2)x^{3n+2}.$$

The left-hand side of this identity is nothing else but

$$g(x^3)f(x) = \left(\sum\nolimits_{n=0}^{\infty} g_{3n+1}x^{3n+1}\right)\left(1-3\sum\nolimits_{n=1}^{\infty} f_n x^n\right).$$

If we define $g_{3n} = 0$, $g_{3n+2} = 0$, $A_{3n} = 0$, $A_{3n+1} = \sigma(3n+1)$, $A_{3n+2} = -\sigma(3n+2)$, we have

$$\left(\sum_{n=0}^{\infty} g_n x^n\right)\left(1-3\sum_{n=1}^{\infty} f_n x^n\right) = \sum_{n=0}^{\infty} A_n x^n$$

and clearly

$$A_n = -3\sum_{k=0}^{n-1} g_k f_{n-k} + g_n.$$

Since we know A_n, we have some interesting equations relating f_m and g_m. Consider first the case where n is congruent to $0 \bmod 3$. The equation for A_n yields

$$0 = \sum_{k=0}^{n-1} g_k f_{n-k}.$$

Since g vanishes when k is congruent to either 0 or $2 \bmod 3$, the sum runs only over those k which are congruent to $1 \bmod 3$. In this case, however, since n is congruent to $0 \bmod 3$, $n-k$ is congruent to $2 \bmod 3$ and thus $f_{n-k} = 0$. Hence there is no information in this case.

Consider now the case n congruent to $2 \bmod 3$. The equation for A_n now yields

$$-\sigma(n) = -3\sum_{k=0}^{n-1} g_k f_{n-k}.$$

The sum still only runs over those k congruent to $1 \bmod 3$ and we get

$$\sigma(3j+2) = 3\sum_{l=0}^{l=j} g_{3l+1} f_{3(j-l)+1}.$$

However, we already know that $f_{3(j-l)+1} = g_{3(j-l)+1}$ so we can write the identity

$$\sigma(3j+2) = 3\sum_{l=0}^{l=j} g_{3l+1}g_{3(j-l)+1}.$$

THEOREM 1:

$$3y^2\prod_{n=1}^{\infty}\frac{(1-y^{9n})^6}{(1-y^{3n})^2} = \sum_{k=0}^{\infty}\sigma(3k+2)y^{3k+2}.$$

Proof: The formula preceding the statement of the theorem without the 3 is precisely the expression for the coefficients of the power series of the function

$$g^2(y^3) = \left(y\prod_{n=1}^{\infty}\frac{(1-y^{9n})^3}{1-y^{3n}}\right)^2$$

and this is the theorem. ∎

THEOREM 2: *For every* $n \in \mathbb{Z}_+$, $g_{3n+1} = \delta(3n+1)$.

Proof: The identity of Theorem 2 can, of course, be rewritten as

$$3\left(\sum_{n=1}^{\infty}\frac{(1-x^{3n})^3}{1-x^n}\right)^2 = \sum_{k=0}^{\infty}\sigma(3k+2)x^k.$$

In [F1] it was shown that the right-hand side is equal to $3(\sum_{n=0}^{\infty}\delta(3n+1)x^n)^2$ and therefore

$$\prod_{n=1}^{\infty}\frac{(1-x^{3n})^3}{1-x^n} = \sum_{n=1}^{\infty}\delta(3n+1)x^n$$

which, by the very definition of g_{3n+1}, proves the theorem. ∎

Let us now combine the identity at the beginning of this section with Theorem 2:

$$3x^2\prod_{n=1}^{\infty}\frac{(1-x^{9n})^6}{(1-x^{3n})^2} + x\prod_{n=1}^{\infty}\frac{(1-x^{9n})^3(1-x^n)^3}{(1-x^{3n})^2} = \sum_{k=0}^{\infty}\sigma(3k+1)x^{3k+1}.$$

After factoring out

$$x\prod_{n=1}^{\infty}\frac{(1-x^{9n})^3}{1-x^{3n}} = g(x^3),$$

we obtain from our representation of $f(x)$ that

$$\left(\sum_{n=0}^{\infty}g_{3n+1}x^{3n+1}\right)\left(1+6\sum_{n=1}^{\infty}\delta(n)x^{3n}\right) = \sum_{k=0}^{\infty}\sigma(3k+1)x^{3k+1}.$$

More compactly, we have

$$\sum_{n=0}^{\infty}(\sigma(3n+1)-g_{3n+1})x^{3n+1}=6\Big(\sum_{n=0}^{\infty}g_{3n+1})x^{3n+1}\Big)\Big(\sum_{n=1}^{\infty}\delta(n)x^{3n}\Big).$$

The formula for the coefficients now yields the same result we would have obtained from the expression for A_n when n is congruent to $1 \bmod 3$:

$$A_{3n+1}=\sigma(3n+1)=\delta(3n+1)+6\sum\nolimits_{j=0}^{n-1}\delta(3j+1)\delta(n-j).$$

4. An alternative proof

In view of our remarks on proofs which use modularity and those which do not, we give in this section a non-modular proof. The reader should not infer from this that we prefer non-modular proofs but rather that we are interested in knowing that one exists when it does. Moreover, we shall see that the added effort in this case is worthwhile, since the computations also provide a solution to a problem of quite a different sort at the end of the paper. The following formula for the product of three theta functions has been derived without any modular considerations ([G]):

$$\theta\begin{bmatrix}\epsilon_1\\ \epsilon_1'\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\epsilon_2\\ \epsilon_2'\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\epsilon_3\\ \epsilon_3'\end{bmatrix}(0,\tau)=$$

$$\sum_{\substack{0\le\mu\le1\\ 0\le\nu\le2}}\theta\begin{bmatrix}\frac{2\mu+\epsilon_1+\epsilon_2}{2}\\ \epsilon_1'+\epsilon_2'\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}\frac{2\nu+\epsilon_1-\epsilon_2+\epsilon_3}{3}\\ \epsilon_1'-\epsilon_2'+\epsilon_3'\end{bmatrix}(0,3\tau)\theta\begin{bmatrix}\frac{6\mu-4\nu+\epsilon_1-\epsilon_2-2\epsilon_3}{6}\\ \epsilon_1'-\epsilon_2'-2\epsilon_3'\end{bmatrix}(0,6\tau).$$

If we substitute here $\epsilon_1=-\frac{1}{3}$, $\epsilon_2=\frac{1}{3}$, $\epsilon_3=-\frac{1}{3}$, $\epsilon_1'=\epsilon_2'=\epsilon_3'=1$, we obtain after some routine theta-algebra

$$\exp\Big(-\frac{\pi i}{3}\Big)\frac{\theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,3\tau)}=\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau)\Big(\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,6\tau)-\theta\begin{bmatrix}\frac{1}{3}\\ 0\end{bmatrix}(0,6\tau)\Big)$$

$$+\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)\Big(\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,6\tau)-\theta\begin{bmatrix}\frac{2}{3}\\ 0\end{bmatrix}(0,6\tau)\Big).$$

The left-hand side is

$$\prod_{n=1}^{\infty}\frac{(1-y^n)^3}{(1-y^{3n})}=f(y)=1-3\sum_{n=1}^{\infty}f_ny^n.$$

The power series version of the right-hand side of this identity is

$$\sum_{n=-\infty}^{\infty} y^{3(n^2+n)}\Big(\sum_{m=-\infty}^{\infty} y^{9m^2+9m+3} - \sum_{m=-\infty}^{\infty} y^{9m^2+3m+1}\Big) + \sum_{n=-\infty}^{\infty} y^{3n^2}\Big(\sum_{m=-\infty}^{\infty} y^{9m^2} - \sum_{m=-\infty}^{\infty} y^{9m^2+6m+1}\Big).$$

We conclude that

$$\begin{aligned}
&3\textstyle\sum_{N=0}^{\infty} f_{3N+1}y^{3N+1}\\
&= \sum_{n=-\infty}^{\infty} y^{3(n^2+n)}\sum_{m=-\infty}^{\infty} y^{9m^2+3m+1} + \sum_{n=-\infty}^{\infty} y^{3n^2}\sum_{m=-\infty}^{\infty} y^{9m^2+6m+1}\\
&= \theta\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}(0,6\tau) + \theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}\frac{2}{3}\\0\end{bmatrix}(0,6\tau),
\end{aligned}$$

$$\begin{aligned}
&1 - 3\textstyle\sum_{N=1}^{\infty} f_{3N}y^{3N}\\
&= \sum_{n=-\infty}^{\infty} y^{3(n^2+n)}\sum_{m=-\infty}^{\infty} y^{9m^2+9m+3} + \sum_{n=-\infty}^{\infty} y^{3n^2}\sum_{m=-\infty}^{\infty} y^{9m^2}\\
&= \theta\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,6\tau) + \theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau).
\end{aligned}$$

Second proof of Theorem 2. By summing on congruence classes modulo 3 in the definition of theta-functions, we get the following identities (see [FK], p. 76):

$$\begin{aligned}
\theta\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau) &= 2\theta\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}(0,18\tau) + \theta\begin{bmatrix}1\\0\end{bmatrix}(0,18\tau),\\
\theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau) &= 2\theta\begin{bmatrix}\frac{2}{3}\\0\end{bmatrix}(0,18\tau) + \theta\begin{bmatrix}0\\0\end{bmatrix}(0,18\tau).
\end{aligned}$$

Hence

$$\begin{aligned}
&\theta\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,6\tau) + \theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau)\\
=&2\left(\theta\begin{bmatrix}1\\0\end{bmatrix}(0,6\tau)\theta\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}(0,18\tau) + \theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau)\theta\begin{bmatrix}\frac{2}{3}\\0\end{bmatrix}(0,18\tau)\right)\\
&+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,6\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,18\tau) + \theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,18\tau),
\end{aligned}$$

i.e.,

$$1 - 3\textstyle\sum_{N=1}^{\infty} f_{3N}y^{3N} = 2(3\sum_{N=0}^{\infty} f_{3N+1}y^{3(3N+1)}) + 1 - 3\sum_{N=1}^{\infty} f_{3N}y^{9N}.$$

By equating the coefficients of $y^{3(3N+1)}$ on both sides we get

$$f_{3(3N+1)} = -2f_{3N+1}.$$

We know, however, that $f_{3N+1} = g_{3N+1}$ and $f_{3(3N+1)} = -2\delta(3N+1)$. Therefore $g_{3N+1} = \delta(3N+1)$, which is precisely Theorem 2. ∎

5. $\delta(N)$ and the partition function

In this section we derive a formula for the function $\delta(n)$ in terms of the classical partition function $P(n)$.

THEOREM 3: *For all $N \in \mathbb{Z}_+$ we have*

(1)
$$\sum_{k\in\mathbb{Z}}^{\frac{3k^2+k}{2}\le N} (-1)^k kP\Big(N - \frac{3k^2+k}{2}\Big) = \delta(N),$$

(2)
$$\sum_{k\in\mathbb{Z}}^{\frac{3k^2+k}{2}\le 3N+1} (-1)^k kP\Big(3N+1 - \frac{3k^2+k}{2}\Big) = \sum_{k=0}^{3\frac{k^2+k}{2}\le N} (-1)^k(2k+1)P\Big(N - 3\frac{k^2+k}{2}\Big),$$

(3)
$$\sum_{k\in\mathbb{Z}}^{\frac{3k^2+k}{2}\le N-1} (-1)^k \delta(N - \frac{3k^2+k}{2}) = \begin{cases} 0 & \text{if } N \ne \frac{3m^2+m}{2}, \\ (-1)^m m & \text{if } N = \frac{3m^2+m}{2}. \end{cases}$$

Proof: The formula

$$\frac{\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)y^{\frac{3n^2+n}{2}}}{\prod_{n=1}^{\infty}(1-y^n)} = 1 + 6\sum_{n=1}^{\infty}\delta(n)y^n$$

and the definition of the partition function yield

$$\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)y^{\frac{3n^2+n}{2}}\sum_{n=0}^{\infty}P(n)y^n = 1 + 6\sum_{n=1}^{\infty}\delta(n)y^n.$$

Euler's identity

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty} x^{\frac{3n^2+n}{2}}$$

yields

$$\sum_{n=0}^{\infty}P(n)y^n\sum_{n=-\infty}^{\infty}(-1)^n y^{\frac{3n^2+n}{2}} = 1,$$

so that we have

$$\left(\sum_{n=-\infty}^{\infty}(-1)^n n y^{\frac{3n^2+n}{2}}\right)\left(\sum_{n=0}^{\infty}P(n)y^n\right)=\sum_{n=1}^{\infty}\delta(n)y^n.$$

We now equate the coefficients of y^N on both sides and get

$$\sum_{k\in\mathbb{Z}}^{\frac{3k^2+k}{2}\leq N}(-1)^k kP\Big(N-\frac{3k^2+k}{2}\Big)=\delta(N),$$

which is (1).

Identity (2) follows immediately by combining (1) with the formula of $g_{3n+1} = \delta(3n+1)$ in the introduction.

The proof of (3) follows from the identity

$$\sum_{n=-\infty}^{\infty}(-1)^n n y^{\frac{3n^2+n}{2}}=\sum_{n=1}^{\infty}\delta(n)y^n\sum_{n=-\infty}^{\infty}(-1)^n y^{\frac{3n^2+n}{2}}. \qquad \blacksquare$$

6. The Diophantine equation $x^2 + 3y^2 = N$

The question of the number of solutions to the equation $x^2 + y^2 = N$ is a classical one, and the quantitative solution was given by Jacobi who proved that the number of solutions was given by $4\Delta(N)$, where $\Delta(N)$ is the non-negative integer $(d_1(N) - d_3(N))$ where $d_i(N)$ represents the number of divisors of N congruent to $i \bmod 4$. In this setting a solution is given by a vector (m, n) and different vectors correspond to different solutions. Hence the number of solutions to $x^2 + y^2 = 1$ is 4 and the solutions are $(1,0), (-1,0), (0,1), (0-1)$. In this section we consider a similar problem. We are interested in the number of solutions to the equation $x^2 + 3y^2 = N$ and will show that if $\delta(N)$ is defined by $d_1(N) - d_2(N)$, then the number of solutions is given by $2\delta(N)$ when N is odd and $6\delta(N)$ when N is even. This is, of course, not a new result, but since it follows from the techniques used here we present it.

LEMMA 2: *The following identity holds true:*

$$\frac{3}{\pi i}\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,3\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,3\tau)}=\theta\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,6\tau)+\theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau).$$

Proof: Let us recall the identity immediately preceding the second proof of Theorem 2 in Section 4:

$$\begin{aligned}1-3\sum\nolimits_{N=1}^{\infty} f_{3N}y^{3N} &= \sum_{n=-\infty}^{\infty} y^{3(n^2+n)} \sum_{m=-\infty}^{\infty} y^{9m^2+9m+3} + \sum_{n=-\infty}^{\infty} y^{3n^2} \sum_{m=-\infty}^{\infty} y^{9m^2} \\ &= \theta\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,6\tau)+\theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau).\end{aligned}$$

However, we know from Section 2 that $f_{3N}=-2\delta(N)$ and that

$$\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)} = 2\pi i\left(\frac{1}{6}+\sum_{n=1}^{\infty}\delta(n)y^n\right).$$

Combining these three results proves the lemma. ∎

In order to proceed we are going to need two further identities: First, the modular equation for $k=3$ (cf. Prop. III.5 in [G] for a proof that doesn't use modularity):

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)=\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau).$$

The second identity we need follows readily from the elementary formulae ([FK], p. 77)

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)&=\theta\begin{bmatrix}0\\0\end{bmatrix}(0,4\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,4\tau),\\ \theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)&=\theta\begin{bmatrix}0\\0\end{bmatrix}(0,4\tau)-\theta\begin{bmatrix}1\\0\end{bmatrix}(0,4\tau).\end{aligned}$$

We write these formulae once more with 3τ substituted for τ, and multiply to get

$$\begin{aligned}&\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,3\tau)\\ &=2\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0,4\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,12\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,4\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,12\tau)\right).\end{aligned}$$

LEMMA 3:

$$\begin{aligned} &3\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)\\ =&2\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0,4\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,12\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,4\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,12\tau)\right)\\ &+\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau).\end{aligned}$$

Proof:

$$\begin{aligned} &3\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)\\ =&2\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)-\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau)\\ &+\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau)\\ =&\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,3\tau)\\ &+\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau)\\ =&2\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0,4\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,12\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,4\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,12\tau)\right)\\ &+\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)+\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau).\end{aligned}$$

(The second equality is the modular equation and the last equality is the identity immediately preceding the statement of the Lemma.) ∎

We now replace τ by 2τ in Lemma 2 and substitute Lemma 1 twice (once as is and once with 4τ instead of τ):

THEOREM 4: *We have the following identity valid for all points τ in the upper half plane:*

$$\pi i\theta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,6\tau)=\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,3\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,3\tau)}+2\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,12\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,12\tau)}.$$

The power series version of Theorem 6 immediately yields

COROLLARY 1: *The number of integer solutions to the equation*

$$x^2 + 3y^2 = N$$

is $2\delta(N)$ when N is odd and $6\delta(N)$ when N is even.

It is perhaps interesting to note that the result of Corollary 1 is equivalent to the following identity, which apparently goes back to Dirichlet as does the result of Corollary 1 (cf. [H], sect. 12.4):

$$\sum_{m,n=-\infty}^{\infty} t^{m^2+mn+n^2} = 1 + 6\sum_{n=1}^{\infty} \delta(n)t^n.$$

References

[BB] J. M. Borwein and P. B. Borwein, *A cubic counterpart of Jacobi's Identity and AGM*, Transactions of the American Mathematical Society **323** (1991), 691–701.

[DKK] D. Dummit, H. Kisilevsky and J. McKay, *Multiplicative products of the η-products*, Contemporary Mathematics **45** (1985), 89–98.

[F1] H. M. Farkas, *On an arithmetical function*, Ramanujan Journal **8** (2004), 309–315.

[F2] H. M. Farkas, *Variations on a theorem of Euler*, Proceedings of the Ashkelon Conference on Complex Function Theory (1996), Israel Mathematical Conference Proceedings **11** (1997), 75–81.

[FK] H. M. Farkas and I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group*, Graduate Studies in Mathematics, Vol. **37**, American Mathematical Society, Providence, RI, 2001.

[G] Y. Godin, *Polynomial identities of theta functions with rational characteristics*, Ph.D. thesis, The Hebrew University of Jerusalem, 2003.

[H] L. K. Hua, *Introduction to Number Theory*, Springer, Berlin, 1982.

[M] Y. Martin, *Multiplicative eta-quotients*, Transactions of the American Mathematical Society **348** (1996), 4825–4856.

[S] L. C. Shen, *On the modular equations of degree 3*, Proceedings of the American Mathematical Society **122** (1994), 1101–1114.